4 Unit Math Homework for Year 12

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2 Topic 2 — Conics Part 4

2.8 Further Geometric Properties of the Ellipse and Hyperbola

Definition:

1. The segment of the tangent to an ellipse or hyperbola between the point of the contact and the directrix subtends a right angle at the corresponding focus. \((PS \perp ST)\)

2. The tangent at a point \(P\) on the ellipse or hyperbola is equally inclined to the focal chords through \(P\). \((\angle SPT = \angle S'PT')\)

3. The normal at a point \(P\) on the ellipse or hyperbola is equally inclined to the focal chords through \(P\).
Example 2.8.1 $P(a \sec \theta, b \tan \theta)$ lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The tangent at $P$ meets the tangents at the ends of the major axis at $Q$ and $R$. Show that $QR$ subtends a right angle at either focus. Deduce that if $P$ is the point $\frac{5}{3}, \frac{4}{3}$ on the hyperbola $\frac{x^2}{9} - y^2 = 1$, with foci $S$ and $S'$, then $Q, S, S, S'$ are concyclic and find the equation of the circle through these points.

**Solution:** Let the tangent at $P$ meet $x = a, x = -a$ in the $Q, R$ respectively.

Let QR meet the $y$-axis in $C$.

Tangent $PR$ has equation $\frac{x \sec \theta}{a} - \frac{y \tan \theta}{b} = 1$.

Hence $Q$ has coordinates $\left[a, \frac{b \tan \theta}{(\sec \theta - 1)}\right]$ and $R$ has coordinates $\left[-a, -\frac{b \tan \theta}{(\sec \theta + 1)}\right]$.

Gradient $QS$. gradient $RS = \frac{b(\sec \theta - 1)}{a \tan \theta(1 - e)} \times \frac{-b(\sec \theta + 1)}{-a \tan \theta(1 + e)}$

$$= \frac{b^2}{a^2(1 - e^2)} \times \frac{(\sec^2 \theta - 1)}{\tan^2 \theta}.$$  

Then $b^2 = a^2(e^2 - 1)$ \Rightarrow gradient $QS \times$ gradient $RS = -1$, \, $\therefore QS \perp RS$  

Similarly, replacing $e$ by $-e$, $QS' \perp RS'$.

Hence $QR$ subtends angles of $90^\circ$ at each of $S$ and $S'$, and $Q, S, R, S'$ are noncyclic, with $QR$ the diameter of the circle through the points.

The $y$-axis is the perpendicular bisector of the chord $SS'$.

Hence the centre of this circle is the point $C$ where the diameter $QR$ meets the $y$-axis.

If $P(\frac{5}{3}, \frac{4}{3})$ lies in the hyperbola $\frac{x^2}{9} - y = 1$, then $QR$ has equation $\frac{5x}{9} - \frac{4y}{3} = 1$ and meets the $y$-axis in $C(0, \frac{2}{3})$.

Also $b^2 = a^2(e^2 - 1)$ gives $e = \frac{10}{9}$, and $S$ ah $s$ coordinates $(\sqrt{10}, \, 0)$.

Hence $CS^2 = \frac{169}{16}$ and the circle through $Q, R, S, S'$ has equation $x^2 + (y + \frac{3}{4})^2 = \frac{169}{16}$.  

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2.9 The Rectangular Hyperbola

2.9.1 Cartesian Equation of the Rectangular Hyperbola

**Definition:** A hyperbola is called rectangular if its asymptotes meet at right angles.

The asymptotes of \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \) have equations \( y = \pm \frac{b}{a} x \).

Hence asymptotes perpendicular \( \Rightarrow \) \( (\frac{b}{a}) \times (-\frac{b}{a}) = -1 \), \( \Rightarrow \) \( a = b \).

Then \( b^2 = a^2(e^2 - 1) \), \( \Rightarrow \) \( e = \sqrt{1 + \frac{b^2}{a^2}} \), \( \Rightarrow \) \( e = \sqrt{2} \).

Hence a rectangular hyperbola, with major axis along the x-axis, has equation \( x^2 - y^2 = a^2 \), eccentricity \( e = \sqrt{2} \) and asymptotes \( y = \pm x \).

2.9.2 Parametric Equations of the Rectangular Hyperbola

**Definition:** The standard parametric from of the rectangular hyperbola \( xy = c^2 \) is

\( x = ct \) and \( y = \frac{c}{t} \),

where the value of parameter \( t \) depends on the position of \( P(ct, \frac{c}{t}) \) on the curve as in the figure shown below:
Exercise 2.9.1

1. For the rectangular hyperbola $xy = 4$, find the parametric equation.

2. For the rectangular hyperbola $x = 4t$, $y = \frac{4}{t}$, find the Cartesian equation.

3. Find the parametric equation of the rectangular hyperbola $xy = 25$.

4. Find the Cartesian equation of the rectangular hyperbola $x = 3t$, $y = \frac{3}{t}$.
2.9.3 Equation of a Chord of the Hyperbola $xy = c^2$

**Cartesian form:**

Let $P(x_1, y_1), Q(x_2, y_2)$ lie on the hyperbola $xy = c^2$.

The gradient of PQ is $\frac{y_2 - y_1}{x_2 - x_1} = \frac{c^2}{x_2^2} - \frac{c^2}{x_1^2} = -\frac{c^2}{x_1x_2}$.

$\therefore$ chord PQ has equation $c^2x + x_1x_2y = k$, $k$ is constant.

$\therefore k = c^2(x_1 + x_2)$.

The chord from $P(x_1, y_1)$ to $Q(x_2, y_2)$ on $xy = c^2$ has equation $c^2x + x_1x_2y = c^2(x_1 + x_2)$.

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**Parametric form:**

Let $P(cp, \frac{c}{p}), Q(cq, \frac{c}{q})$ lie on the hyperbola $xy = c^2$.

The gradient of PQ is $\frac{\frac{c}{q} - \frac{c}{p}}{cp - cq} = -\frac{1}{pq}$.

$\therefore$ chord PQ has equation $x + pqy = k$, $k$ is constant.

$P(cp, \frac{c}{p})$ lies on $PQ \Rightarrow \therefore k = c(p + q)$.

The chord from $P(cp, \frac{c}{p})$ to $Q(cq, \frac{c}{q})$ on $xy = c^2$ has equation $x + pqy = c(p + q)$.

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**Example 2.9.1** The points $P(cp, \frac{c}{p})$ and $Q(cq, \frac{c}{q})$ lie on the rectangular hyperbola $xy = c^2$. The chord PQ subtends a right angle at the another point $R(cr, \frac{c}{r})$ on the hyperbola. Show the the normal at $R$ is parallel to PQ.

**Solution:** The gradient of PR is $\frac{c(\frac{1}{r} - \frac{1}{cp})}{c(p-r)} = -\frac{1}{pr}$.

the gradient of QR is $\frac{c(\frac{1}{r} - \frac{1}{cq})}{c(q-r)} = -\frac{1}{qr}$.

Therefore $PQ \perp QR$, $\Rightarrow$ gradient $PR \times$ gradient $QR = -1$.

$\therefore \frac{1}{prqr} = -1 \Rightarrow r^2 = -\frac{1}{pq}$.

The normal at the point $R(cr, \frac{c}{r})$ has gradient of $r^2$.

the gradient of PQ is $\frac{c(\frac{1}{r} - \frac{1}{pq})}{c(P-q)} = -\frac{1}{pq}$.

Since $r^2 = -\frac{1}{pq}$, then gradient of the normal at R equal to gradient of PQ.

Thus the normal at the point R is parallel to the chord PR.
2.9.4 Equations of Tangents and Normals to the hyperbola \( xy = c^2 \)

Cartesian form of equation of tangent:

By implicit differentiation, \( xy = c^2 \), \( y + x \frac{dy}{dx} = 0 \)

\[ \frac{dy}{dx} = -\frac{y}{x} \]

Gradient of tangent at \( P(x_1, y_1) \) on the hyperbola is \( -\frac{y_1}{x_1} \)

\[ \therefore \text{tangent at } P \text{ has equation } y_1x + x_1y = k, \ k \text{ is constant.} \]

By \( P \) lies on tangent \( \therefore k = 2x_1y_1 = 2c^2. \)

The tangent to \( xy = c^2 \) at \( P(x_1, y_1) \) has equation \( xy_1 + yx_1 = 2c^2 \)

Cartesian form of equation of normal:

Also the normal at \( P(x_1, y_1) \) and equation \( y - y_1 = \frac{x_1}{y_1}(x - x_1). \)

The normal to \( xy = c^2 \) at \( P(x_1, y_1) \) has equation \( xx_1 - yy_1 = x_1^2 - y_1^2. \)

Parametric form of equation of tangent:

\[ x = ct, \ \Rightarrow \ \frac{dx}{dt} = c, \ \text{and } y = \frac{c}{t}, \ \Rightarrow \ \frac{dy}{dt} = -\frac{c}{t^2}, \]

\[ \therefore \frac{dy}{dx} = -\frac{1}{t^2} \]

Gradient of tangent at \( P(cp, \frac{c}{p}) \) on the hyperbola is \( -\frac{1}{p^2}, \)

\[ \therefore \text{tangent at } P \text{ has equation } x + p^2y = k, \ k \text{ is constant.} \]

But \( P \) lies on tangent \( \therefore k = 2cp. \)

The tangent to \( xy = c^2 \) at \( P(cp, \frac{c}{p}) \) has equation \( x + p^2y = 2cp \).

Parametric form of equation of normal:

Also the normal at \( P(cp, \frac{c}{p}) \) has gradient \( P^2 \) and equation \( y - \frac{c}{p} = p^2(x - cp). \)

The normal to \( xy = c^2 \) at \( P(cp, \frac{c}{p}) \) has equation \( px - \frac{1}{p}y = c\left(p^2 - \frac{1}{p^2}\right) \).

Example 2.9.2 Find the equation of the tangent and the normal to the rectangular hyperbola \( x = 2t, \ y = \frac{2}{t} \) at the point where \( t = 4 \).

Solution: For the hyperbola \( x = 2t, \ y = \frac{2}{t} \), where we have \( c = 2. \)

Hence the tangent to the hyperbola \( x = 2t, \ y = \frac{2}{t} \) at point where \( t = 4 \)

has equation \( x + t^2y = 2ct, \ \Rightarrow \ x + 16y = 16 \)

and the normal has equation \( tx - \frac{1}{t}y = c\left(t^2 - \frac{1}{t^2}\right) \Rightarrow 32x - 2y = 255. \)
Exercise 2.9.2

1. Find the equations of the tangent and the normal to the rectangular hyperbola $xy = 12$ at the point (-3, -4).

2. Find the equations of the tangent and the normal to the rectangular hyperbola $x = 3t$, $y = \frac{3}{t}$ at the point $t = -1$.

3. Find the equation of chord of contact of tangent from the point (1, -2) ro the hyperbola $xy = 6$.

4. Find the equation of the tangent and the noram to the rectangular hyperbola $xy = 8$ at the point (4, 2).
2.9.5 Equation of the Chord of Contact of Tangent from an External Point

**Definition:** T lies on the tangent PT and QT, \( \therefore x_0y_1 + y_0x_1 = 2x^2 \) and \( x_0y_2 + y_0x_2 = 2c^2 \).

Hence \( P(x_1, y_1) \) and \( Q(x_2, y_2) \) both satisfy \( x_0y + y_0x = 2c^2 \).

\( \therefore \) this linear equation must be the equation of PQ.

![Diagram showing the chord of contact of tangent from an external point](image)

The chord contact of tangent from an external point \( T(x_0, y_0) \) to the hyperbola \( xy = c \) has equation \( xy_0 + yx_0 = 2c^2 \).

**Example 2.9.3** Find the equation of the chord of contact of tangent from the point \((2, 1)\) to \( xy = 10 \).

**Solution:** For the hyperbola \( xy = 10 \), we have \( c^2 = 10 \).

Hence the chord of contact of tangents from the point \( T(x_0, y_0) = T(2, 1) \)
to the hyperbola \( xy = 10 \) has equation \( xy_0 + yx_0 = 2c^2 \) \( \Rightarrow x + 2y = 20 \).

**Exercise 2.9.3** Find the equation of the chord of contact of tangents from the point \((-1, -3)\) to the rectangular hyperbola \( xy = 4 \). Hence find the coordinated of their points of contact and the equations of these tangents.
2.9.6 Further geometric properties of the rectangular hyperbola

**Definition:**

1. The area of the triangle bounded by a tangent and the asymptotes is a constant.
   
   Let the tangent at $P(ct, \frac{c}{t})$ on $xy = c^2$ meet the x-axes and y-axes in R and T respectively.
   
   The tangent has equation $x + t^2y = 2ct$. At $T, x = 0$, $y = \frac{2c}{t}$.
   
   $\therefore$ Area $\triangle OTR = \frac{1}{2} \times 2ct \times \frac{2c}{t} = 2c^2$.

2. The length of the intercept cut off a tangent by the asymptotes is twice the distance from the point of the intersection of the asymptotes to the point of the contact of the tangent.

   $$TR^2 = \left(\frac{2c^2}{t}\right) + (2ct)^2 (t^2 + \frac{1}{t^2}) \quad \text{and} \quad OP^2 (ct)^2 + \left(\frac{c}{t}\right)^2 = \frac{c^2}{2}(t^2 + \frac{1}{t^2}).$$

   $TR^2 = 4OP^2$, $\Rightarrow \therefore TR = 2OP$.

3. The product of the focal distance of a point $P$ on a rectangular hyperbola is equal to the square of the distance from $P$ to the point of intersection of the asymptotes.

   Let $xy = c^2$ have foci $S$ and $S'$, and directrices $m$ and $m'$. Let $M, M'$ be the feet of the perpendiculars from $P(ct, \frac{c}{t})$ to $m, m'$ respectively.

   Then $PS \times PS' = ePM \times ePM' = 2PM \times PM'$ (since $e = \sqrt{2}$),

   $\therefore$ $PS \times PS' = 2 \times \frac{|ct + \frac{c}{t} - \sqrt{2}c|}{\sqrt{2}} \times \frac{|ct + \frac{c}{t} + \sqrt{2}c|}{\sqrt{2}} = |(ct + \frac{ct}{t})^2| - (\sqrt{2}c)^2$

   $\therefore$ $PS \times PS' = (ct)^2 + (\frac{c}{t})^2 = OP^2$. 
Example 2.9.4 For the rectangular hyperbola $xy = 16$, find (a) the eccentricity; (b) the coordinates of the foci; (c) the equations of the directrices; (d) the equations of the asymptotes. Sketch the hyperbola.

**Solution:** For the hyperbola $xy = 16$ we have $c^2 = 16 \Rightarrow c = 4$.

Hence the hyperbola $xy = 16$ has eccentricity $e = \sqrt{2}$.

Foci: $S(c\sqrt{2}, c\sqrt{2})$, $\Rightarrow S(4\sqrt{2}, 4\sqrt{2})$ and $S'(-c\sqrt{2}, -c\sqrt{2})$, $\Rightarrow S'(-4\sqrt{2}, -4\sqrt{2})$.

Directrices: $x + y = \pm c\sqrt{2}$, $\Rightarrow x + y = \pm 4\sqrt{2}$.

Asymptotes: $x = 0$ and $y = 0$.

Exercise 2.9.4 The point $P(ct, \frac{c}{t})$, where $t = -1$ lies on the rectangular hyperbola $xy = c^2$. The tangent and normal at $P$ meet the line $y = x$ at $T$ and $N$ respectively. Show that $OT \times ON = 4c^2$. 
2.9.7  Locus Problem and the Rectangular Hyperbola

Example  2.9.5  \( P(ct, \frac{c}{t}) \) is a point on the rectangular hyperbola \( xy = c^2 \). \( N \) is the foot of the perpendicular from \( P \) to the x-axis and \( M \) is the midpoint of \( PN \). Find the Cartesian equation of the locus of \( M \) as the position of \( P \) varies. and describe this locus geometrically.

**Solution:**  \( M \) has coordinates \( (ct, \frac{c}{t^2}) \).

Hence the locus of \( M \) has parametric equations \( x = ct \), and \( y = \frac{c}{t^2} \) and Cartesian equation \( xy = \frac{1}{2}c^2 \).

\( M \) traces a rectangular hyperbola as the position of \( P \) varies.

Example  2.9.6  \( P(3p, \frac{3}{p}) \) and \( Q(3q, \frac{3}{q}) \) are point on different branches of the hyperbola \( xy = 9 \). Find the coordinates of the point of intersection \( T \) of the tangents at \( P \) and \( Q \). Find the locus of \( T \) if the positions of \( P \) and \( Q \) vary so that the chord \( PQ \) passes through \((0, 4)\).

**Solution:**  \( T \) lies on the tangent at \( P \) and \( Q \).  \( \Rightarrow \) At \( T \), \( \begin{align*}
  x + p^2y &= 6p \\
  x + q^2y &= 6q
\end{align*} \)

\( (1) - (2) \Rightarrow (p^2 - q^2)y = 6(p - q), \ p \neq q \ \Rightarrow \ \therefore \ \ y = \frac{6}{p+q} \)

Then \( (1) \Rightarrow x = \frac{6pq}{p+q}, \ \Rightarrow \ T \) has coordinates \( \left( \frac{6pq}{p+q}, \frac{6}{p+q} \right) \).

If \((0, 4)\) lies on chord \( PQ \), with equation \( x + pqy = 3(p + q) \), then \( 4pq = 3(p + q) \) and \( T \) is \( T \left( \frac{9}{2}, \frac{9}{2pq} \right) \).

\( P, Q \) on different branches \( \Rightarrow \) \( pq < 0 \), \( \Rightarrow \ \therefore \) locus of \( T \) is \( x = \frac{9}{2}, \ y < 0 \)
Exercise 2.9.5

1. P and Q are variable points on the rectangular hyperbola $xy = 9$. The tangents at P and Q meet at R. If PQ passes through the point (6, 2), find the equation of the locus of R.

2. The point $P(ct, ct)$ lies on the rectangular hyperbola $xy = c^2$. The tangent at P cuts the x-axis at X and the y-axis at Y. Show that the area of $\triangle YOX$ is independent of t.

3. On the rectangular hyperbola $xy = c^2$ there are variable points P and Q. The tangents at P and Q meet at R. Find the equation of the locus of R if PQ passes through the point $(a, 0)$.

4. The point $P(ct, \frac{c}{t})$ lies on the rectangular hyperbola $xy = c^2$. The tangent at P cuts the x-axis at X and the y-axis at Y. Show that PX = PY.
Exercise 2.9.6

1. The point $P(\ct, \frac{c}{t})$ lies on the rectangular hyperbola $xy = c^2$. Show that the normal at $P$ cuts the hyperbola again at the point $Q$ with coordinates $(-\frac{ct}{c^2}, -ct^3)$. Hence find the coordinates of the point $R$ where the normal at $Q$ cuts the hyperbola again.

2. The point $P(\ct, \frac{c}{t})$ lies on the rectangular hyperbola $xy = c^2$. The normal at $P$ meets the $x$-axis at $A$ and the tangent at $P$ meets the $y$-axis at $B$. $M$ is the midpoint of $AB$. Find the equation of the locus of $M$ as $P$ moves on the hyperbola.